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Decoherence of multicomponent symmetrical superpositions of displaced quantum states

V V Dodonov and L A de Souza

Instituto de Física, Universidade de Brasília, Caixa Postal 04455, 70910-900 Brasília, DF, Brazil

E-mail: vdodonov@fis.unb.br

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Abstract

We study the decoherence of superpositions of displaced quantum states of the form $\sum_{k=1}^{N} c_k \hat{D}(\alpha_k)|g\rangle$ (where $|g\rangle$ is an arbitrary 'fiducial' state and $\hat{D}(\alpha)$ is the usual displacement operator) within the framework of the standard master equation for a quantum damped or amplified harmonic oscillator interacting with a phase-insensitive (thermal) reservoir. We compare two simple measures of the degree of decoherence: the quantum purity and the height of the central interference peak of the Wigner function. We show that for $N > 2$ 'mesoscopic' components of the superposition, the decoherence process cannot be characterized by a single decoherence time. Therefore, we distinguish the 'initial decoherence time' and 'final decoherence time' and study their dependence on the parameters α_k and N. We obtain approximate formulae for an arbitrary state $|g\rangle$ and explicit exact expressions in the special case of $|g\rangle = |m\rangle$, i.e., for (symmetrical) superpositions of displaced Fock states of occupation number m. We show that the superposition with a large number of components N and rich 'internal structure' ($m \sim |\alpha|^2$) can be more robust against decoherence than simple superpositions of two coherent states (with $m = 0$), even if the initial decoherence times coincide. Also, we show how initial pure quantum superpositions are transformed into highly mixed and totally classical superpositions in the case of a phase-insensitive amplifier.

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1. Introduction

During more than two decades after the pioneering papers [1], different aspects of the problem of decoherence of 'macroscopic' quantum superpositions (Schrödinger cat states) were considered in numerous publications, references to which can be found in reviews [2–5]. Some theoretical predictions have already been verified in experiments performed by different

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groups [6–10] and new experimental proposals appeared recently [11–13]. Nonetheless, the problem of decoherence continues to attract attention of many researchers, as can be seen, e.g., from recent publications [14–21]. Moreover, this subject is still far from being exhausted, because only the decoherence of the simplest superpositions of *Gaussian packets* (coherent or squeezed states) has been studied in detail until now (for a few exceptions see, e.g., [22–24]). In particular, one of the most frequently considered models of the Schrödinger cat states is based on the notion of *even and odd coherent states* introduced in [25]. These states have the form

$$|\alpha\rangle_{\pm} = \mathcal{N}_{\pm}(|\alpha|)[\hat{D}(\alpha) \pm \hat{D}(-\alpha)]|0\rangle, \tag{1}$$

where $\mathcal{N}_{\pm}(|\alpha|)$ is the normalization factor, $|0\rangle$ is the vacuum state and

$$\hat{D}(\alpha) = \exp(\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}) \tag{2}$$

is the displacement operator. As was shown in [1, 2], the decoherence time for the superposition (1) with $|\alpha|^2 \gg 1$ is inversely proportional to the square of the distance between the components:

$$T_{\text{decoh}} \sim T_{\text{rel}}/(2|\alpha|^2), \tag{3}$$

where T_{rel} is the relaxation time.

On the other hand, one of the aims of the decoherence theory is to understand why quantum interference effects cease to be noticed in the world of macroscopic objects. In this connection we would like to note that macroscopic objects are not only big, but they have many degrees of freedom (as was emphasized in [3]) and possess rather rich internal structures. Consequently, their superpositions can hardly be characterized by a single parameter, such as the distance $|\alpha|$ between the centers of the components. Therefore, it seems interesting to study how the presence of some kinds of 'fine structures' in the components of the superpositions, as well as the number of such components (which can also be big), can influence the decoherence processes.

Such studies not only have an academic interest, but also their results can be useful for various applications, such as quantum information and computation, where the decoherence plays a negative role. Usually, people try to diminish its influence either by replacing a thermal environment with some engineered ones [17, 26–29] or by choosing special initial quantum states [26, 30]. We consider here only the second way, supposing that the decoherence is caused by a coupling with a usual thermal reservoir. But we shall demonstrate that certain multicomponent superpositions with a specific 'fine structure' have a smaller decoherence rate than simple superpositions of two coherent states of the same effective size.

We confine ourselves to a special family of quantum superpositions

$$|\psi\rangle = A^{-1/2}\sum_{k=1}^{N} c_k \hat{D}(\alpha_k)|g\rangle, \tag{4}$$

where $|g\rangle$ is some 'fiducial' state, c_k are constant complex coefficients and A is the normalization factor. If $|g\rangle$ is the vacuum state $|0\rangle$ and $N = 2$, then (4) is a superposition of two coherent states, whose properties are well known. Therefore, our first goal is to see what can happen if one increases the number of components N. The second goal is to see what can happen if one takes more involved fiducial states. For this purpose, we consider in detail the case when $|g\rangle$ coincides with the Fock state $|m\rangle$ (where m is an arbitrary positive integer). The states $\hat{D}(\alpha)|m\rangle$ are known nowadays under the name *displaced number states*. Their properties were studied by many authors (who used sometimes other names) [31], and methods of their generation and reconstruction were considered in [32] (different schemes of generating nonclassical states, including various 'macroscopic superpositions', were discussed, e.g., in [33–45]; some of them were used and verified in experiments [6–8, 46]).

Superpositions of *coherent* states ($m = 0$) on a circle and methods of their generation were studied in [47–57] ('circular states', $\alpha_k = |\alpha|\exp(\mathrm{i}\phi_k)$). Discrete superpositions of coherent states on a straight line were investigated in [51, 58]; methods of their generation were proposed in [59]. More general superpositions of coherent states were studied in [60, 61]. The decoherence of multiple superpositions of coherent states was considered in [62–64]. The generation and amplification of displaced circular states were studied in [65]. Properties of the superpositions (4) with $m \geqslant 1$ and $c_k = 1$ were considered in [23, 66]. The problem of decoherence of these states was considered in [23, 24], but only for small values of parameters m and $|\alpha|$. In contrast, we put emphasis on the case $|\alpha| \gg 1$, because we are interested in the problem of decoherence of 'macroscopic' superposition states.

The choice of superpositions in the form (4) and specifically with $|g\rangle = |m\rangle$ is motivated by two facts. First, such superpositions have a sufficiently rich structure, so they can be used to model states with 'internal degrees of freedom'. Second, these states are still simple enough to allow for an analytical treatment. The decoherence of even and odd superpositions of *two* displaced number states was considered recently in [67], where it was shown that increasing the degree of excitation m for a fixed distance between the components $|\alpha|$ can result in *increasing* the decoherence time, especially for $m \sim |\alpha|^2$. In the present paper, we extend those results by studying how an increase in the number of superposed states N influences the decoherence time(s).

The plan of the paper is as follows. In section 2, we analyze the structure of the Wigner functions of generic superpositions of many displaced copies of the same basic state $|g\rangle$, emphasizing the role of the central interference peak, which is absent in the coordinate probability density. The evolution of the initial pure superposition state (4), governed by the standard master equations (which describes either the attenuation or phase-insensitive amplification processes), is studied in section 3, where explicit expressions are obtained in the special case $|g\rangle = |m\rangle$. Different approaches to the definition of the 'decoherence time' are considered in section 4, where we compare the time evolution of the 'quantum purity' and the height of the central interference peak of the Wigner function. We also show that approximate (but accurate enough in the case $|\alpha_k| \gg 1$) expressions for the height of the central peak can be found for a rather arbitrary fiducial state $|g\rangle$. This is a serious argument in favor of using the height of the central interference peak as a good indicator of decoherence. A comparison of the dependence of the 'initial' and 'final' decoherence times on the parameters α, m and N for superpositions of displaced number states on a straight line is made in section 5. A transformation of initial *quantum* superpositions to *classical* ones in the case of phase-insensitive amplification is demonstrated in section 6. Section 7 contains conclusions.

2. Wigner functions of superpositions of displaced states

If absolute values of differences between the displacement parameters $\alpha_k - \alpha_j$ are big enough, then the components of the superposition state (4) practically do not overlap. In such a case, how one could see that the state concerned is not a classical mixture, but a quantum *superposition*? The best way is to analyze, instead of the probability density $|\psi(x)|^2$ in the coordinate space, the *Wigner function* (we assume $\hbar \equiv 1$)

$$W(q, p) = \int \mathrm{d}v\, \mathrm{e}^{\mathrm{i}pv} \langle q - v/2|\hat{\rho}|q + v/2\rangle, \tag{5}$$

where $\langle x|\hat{\rho}|x'\rangle$ is the matrix element of the statistical operator $\hat{\rho}$ in the coordinate basis. In order to simplify formulae, we shall also use the function $F(z) = \pi^{-1}W(q, p)$ of the complex argument $z = (q + \mathrm{i}p)/\sqrt{2}$ (assuming the unit mass and frequency of the quantum oscillator under consideration). Then the normalization conditions read

$$\int W(q, p)\,\mathrm{d}q\,\mathrm{d}p/(2\pi) = \int F(z)\,\mathrm{d}^2z = \mathrm{Tr}\,\hat{\rho} = 1, \tag{6}$$

with $\mathrm{d}^2z \equiv \mathrm{dRe}(z)\mathrm{d\,Im}(z)$.

Suppose that the Wigner function $F_g(z)$ of some arbitrary state $|g\rangle$ is known, then one can verify that the Wigner function of the superposition state (4) has the form

$$F_\psi(z) = A^{-1}\sum_{j,k=1}^{N} c_j c_k^* F_g\big(z - \alpha_{jk}^{(+)}\big)\exp\big(2\mathrm{i}\,\mathrm{Im}\big[2\alpha_{jk}^{(-)}z^* - \alpha_{jk}^{(-)}\alpha_{jk}^{(+)*}\big]\big), \tag{7}$$

where

$$\alpha_{jk}^{(\pm)} = \tfrac{1}{2}(\alpha_j \pm \alpha_k) = \pm\alpha_{kj}^{(\pm)}. \tag{8}$$

Function (7) is real if the Wigner function $F_g(z)$ is real. Each of the N^2 terms in the sum (7) has the same form as the Wigner function of the 'fiducial' state $|g\rangle$, but it is displaced by the complex value $\alpha_{jk}^{(+)}$ in the phase plane and multiplied by a z-dependent phase factor (except for the diagonal terms with $j = k$). The non-diagonal terms arise due to the quantum interference (they do not appear for classical mixtures). These interference terms are the most pronounced for an *even* number of components N, if all displacement parameters α_k can be divided into $N/2$ pairs $(\alpha_k, -\alpha_k)$ with equal phases of amplitudes c_k in each component of the pair, because in such a case there exists a big interference peak at the central point $z = 0$. For this reason, we shall study in detail the decoherence properties of this special family of superposition states, assuming that

$$\alpha_{2k-1} = -\alpha_{2k}, \qquad c_{2k-1} = c_{2k}, \qquad 1 \ll |\alpha_2| \leqslant |\alpha_4| \leqslant \cdots \leqslant |\alpha_N| \tag{9}$$

and $|\alpha_k \pm \alpha_j| \gg R$ (obviously, for $k \neq j$ in the case of 'minus' sign), where R is an effective size of the state $|g\rangle$ in the phase plane, so that $|F_g(z)| \ll |F_g(0)|$ if $|z| > R$. Under these conditions, we can write the normalization factor and the heights of the central interference peak (at $z = 0$) and the 'constituent' peaks (at $z = \alpha_k$) as

$$A = \sum_{k=1}^{N}|c_k|^2, \qquad F_\psi(0) = F_g(0), \qquad F_\psi(\alpha_k) = F_g(0)|c_k|^2/A, \tag{10}$$

with small corrections of the order of $\exp(-|\alpha_k \pm \alpha_j|^2)$. Consequently, in the case of equal coefficients $|c_k|^2$, the height of the central interference peak is N times bigger than the heights of the constituent ones.

The existence of the high central interference peak is the best manifestation of the superposition nature of the quantum state. Moreover, the value of the Wigner function at the origin *can be measured*, and it has a clear physical meaning [13, 68]. On the other hand, the structure of the central peak is very sensitive to any perturbations of the quantum state due to the presence of strongly oscillating terms with $j \neq k$ in equation (7). For this reason, studying the time evolution of the central part of the Wigner function one can obtain the information on the decoherence of quantum superpositions in the most simplest way (see also [69, 70]).

In the case of the fiducial Fock state $|m\rangle$, one should put in equation (7) the function [71]

$$F_m(z) = (2/\pi)(-1)^m\exp(-2|z|^2)L_m(4|z|^2), \tag{11}$$

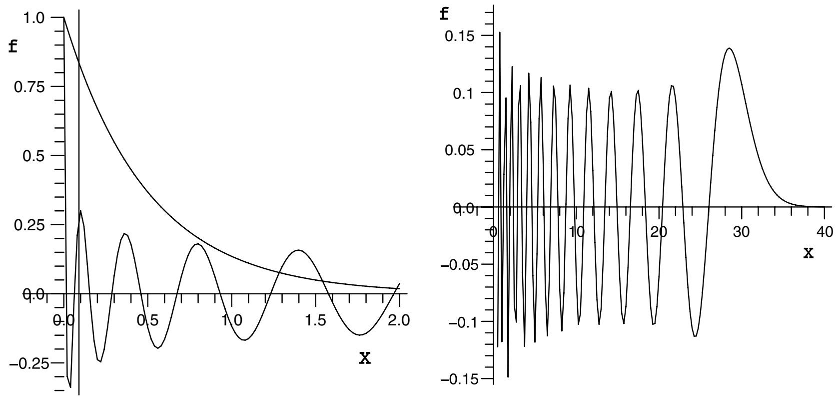

Figure 1. Left: the function $f(x) = \exp(-2x)L_m(4x)$ with $m = 30$ versus the function $\exp(-2x)$ (the upper line) in the interval $x < 2$. Right: the same function in the interval $x > 1$.

where $L_m(x) \equiv L_m^{(0)}(x)$ is the Laguerre polynomial defined according to [72]

$$L_m^{(k)}(z) = \sum_{n=0}^{m} \frac{(m+k)!}{(m-n)!n!(k+n)!}(-z)^n. \tag{12}$$

The combination of (7) and (11) results in a generalization of formulae found in [60] for the superpositions of coherent states ($m = 0$) and in [66] for m arbitrary (but in the special case of $c_k = 1$ and $|\alpha_k| = \text{const}$ with uniformly distributed phases of α_k).

The function $f_m(x) = \exp(-2x)L_m(4x)$ is shown in figure 1. One can see that it has a rather rich structure if $m \gg 1$, and it is interesting to study how this structure influences the decoherence time. It is worth mentioning a significant difference in the 'effective width' of the function $f_m(x)$ in the cases $m = 0$ and $m \gg 1$: while $f_0(x)$ rapidly goes to zero for $x > 1$, for $m \gg 1$ this occurs only if $x > m$. Consequently, function (11) decreases exponentially in the 'classically forbidden region' $|z|^2 > m$, where the energy of oscillator exceeds the value $m\hbar\omega$ (remember that $|z|^2 = (q^2 + p^2)/2$).

3. Time evolution of the Wigner function

We assume that an irreversible evolution of the quantum state in the interaction picture (where rapid oscillations at the oscillator eigenfrequency are eliminated) is governed by the standard master equation [73, 74] describing the influence of a phase-insensitive (in particular, thermal) reservoir:

$$\frac{\partial \rho}{\partial t} = \eta\nu_1 \left(2\hat{a}\rho\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\rho - \rho\hat{a}^\dagger\hat{a}\right) + \eta\nu_2 \left(2\hat{a}^\dagger\rho\hat{a} - \hat{a}\hat{a}^\dagger\rho - \rho\hat{a}\hat{a}^\dagger\right). \tag{13}$$

Here, $\hat{a}$ and $\hat{a}^\dagger$ are the usual boson annihilation and creation operators, respectively. The positive parameter η is proportional to the coupling coefficient between the oscillator (or the chosen field mode) and the reservoir. The parameters ν_1 and ν_2 can be interpreted, e.g., as the numbers of atoms of the reservoir (resonantly interacting with the oscillator) in the fundamental and excited states, respectively. The case $\nu_1 > \nu_2$ describes a phase-insensitive attenuator, while the case $\nu_2 > \nu_1$ corresponds to a phase-insensitive amplifier.

The time-dependent Wigner function $W(q, p, t)$ obeys the Fokker–Planck equation which follows immediately from (13)

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial q}(\gamma q W) + \frac{\partial}{\partial p}(\gamma p W) + D\left(\frac{\partial^2 W}{\partial q^2} + \frac{\partial^2 W}{\partial p^2}\right), \tag{14}$$

where

$$\gamma = \eta\sigma\nu_0, \qquad D = \frac{1}{2}\eta\nu_0 = \frac{\gamma}{2\sigma}, \qquad \nu_0 = \nu_1 + \nu_2, \qquad \sigma = \frac{\nu_1 - \nu_2}{\nu_1 + \nu_2}. \tag{15}$$

The asymmetry parameter σ can vary in the interval $[-1, 1]$, being positive for attenuators and negative for amplifiers. Introducing the reservoir temperature T by means of the relation $\nu_2/\nu_1 = \exp(-\Delta E/\kappa T)$ (where ΔE can be interpreted as the energy difference between the excited and ground levels of atoms in the reservoir and κ is the Boltzmann constant), one can write $\sigma = \tanh(\Delta E/(2\kappa T))$.

The solution to equation (14) can be written as

$$F(z; t) = \int \mathcal{K}(z; t|z', 0) F(z'; 0)\, \mathrm{d}^2 z', \tag{16}$$

where the propagator $\mathcal{K}(z; t|z'; 0)$ can be calculated by means of different methods [4, 75]. Transforming its explicit form given in [4, 76] from (q, p) to z variables (and taking into account that $\mathrm{d}q\,\mathrm{d}p = 2\mathrm{d}^2 z$), we obtain

$$\mathcal{K}(z; t|z'; 0) = \frac{2}{\pi\tau(t)} \exp\left(-\frac{2}{\tau(t)}|z - G(t)z'|^2\right), \tag{17}$$

where

$$G(t) \equiv \exp(-\gamma t) = \sqrt{1 - \sigma\tau}, \tag{18}$$

$$\tau(t) \equiv [1 - G^2(t)]/\sigma, \qquad t(\tau) \equiv -(2\gamma)^{-1} \ln(1 - \sigma\tau). \tag{19}$$

Note that $\tau(t) \geqslant 0$ independently of the sign of parameter σ. In the limit $t \to 0$, we have

$$\tau \approx 2\eta\nu_0 t = 4Dt = 2\gamma t/\sigma,$$

so the parameter τ does not depend on σ (i.e., on the effective temperature) if the diffusion coefficient D is fixed. However, for the fixed damping coefficient γ the usual time t can be much less than τ if $|\sigma| \ll 1$ (i.e., in the high-temperature case). For attenuators we have $G(t) \leqslant 1$ and for amplifiers $G(t) \geqslant 1$.

If $\sigma > 0$ (a thermal bath with non-negative temperature), then $G(t) \to 0$ for $t \to \infty$, so that $\mathcal{K}(z; \infty|z'; 0)$ does not depend on z'. This means that any initial Wigner function goes asymptotically to the equilibrium Wigner function:

$$F_{\mathrm{eq}}(z) = (2\sigma/\pi) \exp(-2\sigma|z|^2). \tag{20}$$

Consequently, all individual features of function (7), including its 'fine structure' shown in figure 1, disappear for $t \gg t_{\mathrm{th}} = \gamma^{-1}$. Thus, t_{th} can be called the *thermalization time*. This time does not depend on the initial quantum state.

However, there are other time scales, of the order of $t_{\mathrm{th}}/|\alpha_k|^2 \ll t_{\mathrm{th}}$, which correspond to the process of *decoherence*, i.e., a fast transformation of the initial *pure* quantum state to a classical *mixture* and disappearance of quantum interference effects. These scales are very sensitive to the initial state.

Applying the propagator (17) to the initial function (7) with $F_g(z)$ given by (11) and using the formula (which is a 'complex' version of the formula derived in [67])

$$\int \mathrm{d}^2 z \exp(-g|z|^2 + \xi z + \eta z^*) L_n(a|z|^2) = \frac{\pi(g-a)^n}{g^{n+1}} \exp\left(\frac{\xi\eta}{g}\right) L_n\left(\frac{a\xi\eta}{g(g-a)}\right), \tag{21}$$

we obtain after some algebra the explicit form of the time-dependent Wigner function for the initial superposition of displaced number states:

$$F(z;t) = \frac{2(-1)^m r^m}{A\pi s^{m+1}} \sum_{j,k=1}^{N} c_j c_k^* L_m \left(\frac{4}{rs}[Gz - \gamma_{jk}][Gz - \chi_{jk}]^* \right)$$
$$\times \exp\left[-\frac{2}{s}|z|^2 - \frac{r}{s}\alpha_j\alpha_k^* + \frac{2G}{s}\left(z\alpha_k^* + z^*\alpha_j\right) - \frac{1}{2}(|\alpha_j|^2 + |\alpha_k|^2) \right], \tag{22}$$

where

$$\gamma_{jk} = \tfrac{1}{2}(r\alpha_j + s\alpha_k), \qquad \chi_{jk} = \tfrac{1}{2}(s\alpha_j + r\alpha_k), \tag{23}$$

$$r(t) = G^2 - \tau \equiv 1 - \tau(1+\sigma), \qquad s(t) = G^2 + \tau \equiv 1 + \tau(1-\sigma). \tag{24}$$

An equivalent (but less compact) expression was obtained in [24] for displaced number states uniformly distributed along the circle $|\alpha_k| = \text{const}$ with equal coefficients $c_k = 1$.

4. Decoherence times

To define and calculate the decoherence time, we need some function $\mathcal{C}(t)$ of a single time variable, which could serve as an adequate *simple* indicator of the 'degree of coherence' of the quantum state. Let us normalize such a function by the condition $\mathcal{C}(0) = 1$, supposing that $\mathcal{C}(t) \leqslant 1$ for $t > 0$. There are at least two simple possibilities to define the decoherence time. One can choose some value $\eta < 1$ to define the 'η-decoherence' time t_η as a solution to the equation

$$\mathcal{C}(t_\eta) = \eta. \tag{25}$$

Another possibility is to define the 'initial decoherence time' (IDT) t_{in} simply as the inverse time derivative of $\mathcal{C}(t)$ at $t = 0$:

$$t_{\text{in}}^{-1} = |\mathrm{d}\mathcal{C}/\mathrm{d}t|_{t=0}. \tag{26}$$

A technical advantage of definition (26) is that the initial derivative of function $\mathcal{C}$ can be found in some cases without any knowledge of $\mathcal{C}(t)$ for $t > 0$ (see illustrations in the following subsections). In the known examples related to the decoherence of simple initial superpositions of *two coherent states*, both the definitions were, in fact, equivalent, because in these examples the functions $\mathcal{C}(t)$ were close to *exponential functions*. In this specific case, each of the two times can be reduced to another by simple rescaling: $t_{\text{in}} = t_{\eta=1/\mathrm{e}}$, $t_\eta = t_{\text{in}} \ln(1/\eta)$. One of the goals of this paper is to show that for superpositions with more than two components N or with a high 'level of excitation' m of each component the definition (26) ceases to characterize the decoherence process correctly, so that one should use the definition (25) with an appropriate choice of the 'tolerant coherence level' η. In the following subsections, we consider and compare two simple functions $\mathcal{C}$: the quantum purity and the height of the central interference peak.

4.1. The quantum purity as an indicator of decoherence

One of the most frequently used simple indicators of coherence is the *quantum purity* $\mu \equiv \mathrm{Tr}(\hat{\rho}^2)$ [14, 21, 77, 78], which can be calculated by means of the formula

$$\mu = \pi \int F^2(z)\, \mathrm{d}^2 z. \tag{27}$$

The initial value of the purity is $\mu(0) = 1$, and for $t \gg t_{\text{th}}$ it goes to the equilibrium value $\mu_{\text{eq}} = \sigma$ (if $\sigma > 0$). But before this happens, it rapidly (during the time of the order of $t_{\text{th}}/|\alpha_k|^2$) goes to the intermediate value

$$\mu_{\text{int}} = \sum_{k=1}^{N} d_k^2, \tag{28}$$

where

$$d_k = |c_k|^2 \Big/ \sum_{k=1}^{N} |c_k|^2, \qquad \sum_{k=1}^{N} d_k = 1. \tag{29}$$

Formula (28) corresponds to the purity of a mixture of N orthogonal pure states. In our case it is approximate, as a matter of fact, because the components of the superposition (4) are not exactly orthogonal, but the corrections are exponentially small, of the order of $\min\{\exp(-|\alpha_k|^2), \exp(-|\alpha_k - \alpha_j|^2)\}$ $(k \neq j)$, if $|\alpha_k| \gg 1$. The purity of each component is approximately preserved at the time scales of the order of $t_{\text{th}}/|\alpha_k|^2$, because at these time scales we can neglect the change of functions $s(\tau)$ and $r(\tau)$ in equation (22), replacing them by the unit values.

The integral (27) with function (22) can be calculated only for $m = 0$ (superpositions of *coherent states* at $t = 0$). Using formula (21) with $n = 0$, we obtain

$$\mu = \sum_{j,k,m,n=1}^{N} \frac{g_{jk} g_{mn}}{A^2 s} \exp\left[\frac{\tau}{s}(\alpha_j - \alpha_m)(\alpha_n - \alpha_k)^*\right], \tag{30}$$

where

$$g_{jk} = c_j c_k^* \exp\left[\alpha_j \alpha_k^* - \tfrac{1}{2}(|\alpha_j|^2 + |\alpha_k|^2)\right]. \tag{31}$$

Since we suppose that $|\alpha_j - \alpha_k| \gg 1$ for $j \neq k$, only the terms with $j = k$ and $m = n$ give significant contributions to the sum in equation (30) for $t \ll t_{\text{th}}$. Thus, we can simplify equation (30) as (here we put $s = 1$)

$$\mu = \sum_{k,m=1}^{N} d_k d_m \exp(-\tau |\alpha_k - \alpha_m|^2), \qquad \tau \ll 1, \tag{32}$$

where the coefficients d_k were defined in equation (29). Equation (32) clearly shows that for superpositions of more than two coherent (mesoscopic) states, there can exist *more than one* decoherence times.

For symmetrical superpositions obeying conditions (9), equation (32) takes the form

$$\mu = 2 \sum_{k,m=1}^{N/2} d_{2k} d_{2m} [\exp(-\tau |\alpha_{2k} - \alpha_{2m}|^2) + \exp(-\tau |\alpha_{2k} + \alpha_{2m}|^2)], \tag{33}$$

and for $\tau \ll 1$ we obtain

$$\mu = 1 - 4\tau \sum_{k=1}^{N/2} d_{2k} |\alpha_{2k}|^2 + \mathcal{O}(\tau^2), \tag{34}$$

so that the 'initial decoherence time' (26) is given by the relations

$$\tau_{\text{in}}^{-1} = |\mathrm{d}\mu/\mathrm{d}\tau|_{\tau=0} = 4 \sum_{k=1}^{N/2} d_{2k} |\alpha_{2k}|^2. \tag{35}$$

The right-hand side of (35) is proportional to the mean number of quanta $\langle\hat{a}^\dagger\hat{a}\rangle$ in the initial state. It can be interpreted also as twice the mean square radius of the superposition in the complex z-plane. The advantage of the IDT based on the quantum purity is that it can be calculated for an arbitrary initial state, even when integrals (16) and (27) cannot be calculated analytically. Indeed, an immediate consequence of equation (13) is the formula [67, 77, 78]

$$\dot{\mu}|_{t=0} = 2\mathrm{Tr}(\hat{\rho}\dot{\hat{\rho}})_{t=0} = -2\eta\nu_0(1-\sigma+2[\langle\hat{a}^\dagger\hat{a}\rangle - |\langle\hat{a}\rangle|^2]_{t=0}). \tag{36}$$

For the initial superpositions of displaced number states, the average values at $t=0$ can be calculated for any m with the aid of the known relations

$$\hat{a}\hat{D}(\alpha) = \hat{D}(\alpha)(\hat{a}+\alpha), \qquad \hat{a}^\dagger\hat{D}(\alpha) = \hat{D}(\alpha)(\hat{a}^\dagger+\alpha^*),$$
$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \qquad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle,$$
$$\hat{D}^\dagger(\alpha_k)\hat{D}(\alpha_j) = \hat{D}(\alpha_j-\alpha_k)\exp[\mathrm{i}\,\mathrm{Im}(\alpha_k^*\alpha_j)],$$

together with formulae for the matrix elements $\langle m|\hat{D}(\gamma)|n\rangle \equiv D_{mn}^{(\gamma)}$ [79]

$$D_{mn}^{(\gamma)} = \begin{cases} \sqrt{\dfrac{m!}{n!}}(-\gamma^*)^{n-m}\,\mathrm{e}^{-|\gamma|^2/2}L_m^{(n-m)}(|\gamma|^2), & n \geqslant m, \\ \sqrt{\dfrac{n!}{m!}}\gamma^{m-n}\,\mathrm{e}^{-|\gamma|^2/2}L_n^{(m-n)}(|\gamma|^2), & m \geqslant n. \end{cases}$$

It can be shown [67] that $|\exp(-x/2)L_m^{(k)}(x)| \ll 1$ for $k=0,1$ and $x \gg 1$, independently of the value of index m (even if $m\sim x$ or $m \gg x$). Therefore, calculating double sums for the average values, we retain only the diagonal terms with $\alpha_j=\alpha_k$. This results in the following generalization of formula (35) under the conditions (9):

$$\tau_{\mathrm{in}}^{-1} \approx 2m + 4\sum_{k=1}^{N/2} d_{2k}|\alpha_{2k}|^2. \tag{37}$$

The neglected terms show some oscillating behavior as a function of m if $m>|\alpha_1|^2$. But the maximal amplitude of these oscillations is of the order of $|\alpha|^{4/3}$ if $m\sim|\alpha|^2$, as was shown in [67]. Consequently, these oscillations do not change the IDT significantly.

However, the IDT cannot be a universal measure of the rate of decoherence, because τ_{in} can be made very small simply by increasing the greatest value $|\alpha_N|^2$, without changing the values of all other coefficients, even if $d_N \ll d_k$ with $k\neq N$. Obviously, the last member of the superposition is quite insignificant in this case, and nothing really happens with the quantum state after the time τ_{in}. This is illustrated in figure 3.

The 'final decoherence time' (FDT) τ_f can be defined as the time when the function $\mu(t)$ becomes close to the value (28), for example, as the solution of equation

$$[\mu(\tau_f)-\mu_{\mathrm{int}}]/\mu_{\mathrm{int}} = \epsilon, \tag{38}$$

where $\epsilon \ll 1$ is some fixed small number and μ_{int} is given by formula (28). Requiring that $\tau_f=\tau_{\mathrm{in}}$ for $N=2$ and $m=0$ (i.e., superpositions of two coherent states), we obtain the value $\epsilon=\mathrm{e}^{-2}\approx 0.135$. However, for $N\gg 1$ or $m>0$ and the same value of ϵ we have $\tau_f \gg \tau_{\mathrm{in}}$, as shown in the examples of section 5.

4.2. *The decay rate of the central interference peak*

Another good indicator of decoherence, at least for symmetrical superpositions satisfying the conditions (9), is the height of the *central interference peak* at $z=0$ [67, 69, 70]. As was shown in section 2, the initial height $F(0, 0)$ can be N times bigger than the height of any constituent peak $F(\alpha_j, 0)$ (if all coefficients c_j have approximately equal values). Moreover, the value $F(0, 0)$ does not depend on the positions α_k of the centers of constituent peaks. It is reasonable to use the normalized height, dividing $F(0, \tau)$ by the initial value. Thus, we arrive at the following indicator of coherence:

$$f(\tau) = \frac{F(z=0;\tau)}{F(z=0;0)}. \tag{39}$$

For symmetrical superpositions of displaced number states, whose parameters satisfy the conditions (9), we obtain the following explicit formula for the normalized height of the central interference peak:

$$f(\tau) = \frac{2r^m}{s^{m+1}} \sum_{k=1}^{N/2} d_{2k} \exp\left[-\frac{2\tau}{s}|\alpha_{2k}|^2\right] L_m\left(-\frac{4\tau^2}{rs}|\alpha_{2k}|^2\right), \tag{40}$$

where the coefficients d_{2k} were defined in (29) and functions $s(\tau)$ and $r(\tau)$ were defined in (24). Equation (40) shows that the evolution of the height of the central interference peak is *non-exponential* in a generic case of more than two components of the superpositions. For this reason, it is difficult (or impossible) to introduce a unique 'decoherence time' in a generic case. The IDT τ_{in} gives only the time of disappearance of the interference between the most distant components of the superposition. But this parameter is insufficient to characterize the whole process, especially if $N \gg 1$ or $d_N \ll 1$. The interference pattern disappears totally only after the 'final decoherence time' $\tau_f \sim |\alpha_1|^{-2}$ (provided d_1 is not much smaller than the other coefficients d_k), which can be much greater than τ_{in}. In particular, increasing the 'size' of the superposition state by increasing $|\alpha_N|$ (or adding new components with bigger displacement parameters) does not affect the value of τ_f.

Looking at the *linear* term (with respect to τ) of the Taylor expansion

$$f(t) \approx 1 - \tau\left[2m + 1 - \sigma + 4\sum_{k=1}^{N/2} d_{2k}|\alpha_{2k}|^2\right] + \tau^2\left[4\sum_{k=1}^{N/2} d_{2k}|\alpha_{2k}|^2(|\alpha_{2k}|^2 + 4m + 2 - 2\sigma) + 2m^2 + 2m - 4m\sigma + (1-\sigma)^2\right], \tag{41}$$

we see that the *initial* 'f-decoherence time', defined as $\tau_{\text{in}}^{-1} = |\mathrm{d}f/\mathrm{d}\tau|_{\tau=0}$, coincides with the '$\mu$-decoherence time' (37) (where the term $1 - \sigma$ was neglected). As a matter of fact, this coincidence takes place for arbitrary superpositions, provided they have a definite parity with respect to the reflection $x \to -x$. Indeed, the quantity $\partial W/\partial t|_{t=0}$ can be found directly from the Fokker–Planck equation (14) without any knowledge of function $W(q, p, t)$ for $t > 0$. On the other hand, Wigner functions of pure quantum states with definite parity have a remarkable property that their derivatives at the origin of the phase plane (q, p) are determined completely by the mean values of powers and products of the canonical operators [80]. In particular,

$$\frac{1}{W}\frac{\partial^2 W}{\partial q^2}\bigg|_{q=p=0} = -4\langle\hat{p}^2\rangle, \qquad \frac{1}{W}\frac{\partial^2 W}{\partial p^2}\bigg|_{q=p=0} = -4\langle\hat{q}^2\rangle. \tag{42}$$

Putting these expressions on the right-hand side of equation (14) at $t = 0$ with account of (15) and (18), we obtain the derivative $\mathrm{d}f/\mathrm{d}\tau|_{\tau=0} = \sigma - \langle\hat{p}^2\rangle - \langle\hat{q}^2\rangle$, which coincides exactly with (36), due to the relations $\langle\hat{p}^2\rangle + \langle\hat{q}^2\rangle = 2\langle\hat{a}^\dagger\hat{a}\rangle + 1$ and $\langle\hat{a}\rangle = 0$.

We see that the IDT decreases with the increase in the parameter m. On the other hand, the *quadratic* term in (41) indicates that by increasing m one slows down the decay of the function $f(\tau)$. Consequently, the states with $m > 0$ can have *bigger* 'final decoherence times' than the states with $m = 0$. Such a behavior becomes especially clear for $m \gg 1$, since formula (41) shows that the influence of parameter m is significant for $m \sim |\alpha|^2 \gg 1$. In this case, one can replace the Laguerre polynomial of the negative argument by the Hilb asymptotical formula [72]

$$L_m(-x) \sim \exp(-x/2) I_0(\sqrt{4mx}), \tag{43}$$

where $I_0(z)$ is the modified Bessel function. Formula (43) holds uniformly with respect to x (including small values) for $m \gg 1$. Then, equation (40) takes the form

$$f(\tau) \approx \frac{2r^m}{s^{m+1}} \sum_{k=1}^{N/2} d_{2k} \exp\left(-\frac{2\tau}{rs} G^2 |\alpha_{2k}|^2\right) I_0\left(\frac{4\tau|\alpha_{2k}|\sqrt{m}}{\sqrt{rs}}\right). \tag{44}$$

For $\tau \ll 1$ and $m \gg 1$, the ratio r^m/s^{m+1} can be represented with a sufficient accuracy as $\exp(-2m\tau)$ (the terms of the order of $m\tau^2$ in the argument of the exponential function can be obviously neglected). On the other hand, if the argument of the Bessel function $4\tau|\alpha_{2k}|\sqrt{m}$ is significantly greater than unity (this just occurs in the regime of 'final decoherence', when the function $f(\tau)$ must be small), then one can use the asymptotical formula for the modified Bessel function $I_0(x) \approx (2\pi x)^{-1/2} \exp(x)$. Moreover, we can still put $s(\tau) = r(\tau) = G(\tau) = 1$ in the arguments of the exponential and Bessel function (corrections are of the order $|\alpha|^2\tau^2 \ll 1$, if $|\alpha|^2\tau \sim 1$). Then, (44) can be written as

$$f(\tau) \approx \sum_{k=1}^{N/2} \frac{d_{2k} \exp[-2\tau(|\alpha_k| - \sqrt{m})^2]}{(2\pi\tau|\alpha_k|\sqrt{m})^{1/2}}. \tag{45}$$

Numerical tests show no visible difference between the functions (44) and (45) for $|\alpha|^2\tau > 2$ and $m/|\alpha|^2 > 1/3$. According to equation (45), the rate of decreasing the central interference peak becomes essentially slow if one of the values of $|\alpha_k|$ is close to $\sqrt{m}$. For such superpositions, the final decoherence time can be much bigger than τ_{in}. These features are illustrated in section 5.

An *approximate* formula for the time-dependent height of the central interference peak for a general superposition (4) can be found, if one puts $z = 0$ in equations (16) and (17) and takes into account only those terms in the double sum which give $\alpha_{jk}^{(+)} = 0$. For the states satisfying the conditions (9) this means that $\alpha_{jk}^{(-)} = \alpha_j$. Thus, we obtain (changing the integration variable $z' = y\sqrt{\tau}/G\sqrt{2}$) the following expression for the normalized height:

$$f(\tau) = \sum_{k=1}^{N/2} \frac{2d_{2k}}{\pi G^2 F_g(0)} \text{Re}\left\{\int \mathrm{d}^2 y F_g\left(\frac{y\sqrt{\tau}}{G\sqrt{2}}\right) \exp\left[-|y|^2 + \frac{\sqrt{2\tau}}{G}\left(\alpha_{2k} y^* - \alpha_{2k}^* y\right)\right]\right\}, \tag{46}$$

where the coefficients d_{2k} were defined in (29). The main contribution to the integrals comes from the domain $|y| < Y \sim 1$ (say, $Y = 3$, since $\mathrm{e}^{-9} \approx 10^{-4} \ll 1$). But studying the decoherence process we are interested in the time scale $\tau \ll 1$. Consequently, confining ourselves by symmetrical fiducial states, whose Wigner functions have an extremum at point $z = 0$, we can replace the exact function $F_g(z)$ under the integral by its Gaussian approximation

$$F_g(z) \approx F_g(0) \exp(-4\sigma_p[\text{Re}(z)]^2 - 4\sigma_q[\text{Im}(z)]^2), \qquad |z| \ll 1, \tag{47}$$

where σ_q and σ_p are the coordinate and momentum variances, according to formula (42), which is totally applicable in the case concerned (we choose the axes in the complex z-plane in such a way that the cross term $\text{Re}(z)\text{Im}(z)$ disappears; also we remember that $\langle\hat{q}\rangle = \langle\hat{p}\rangle = 0$ for symmetrical states). Then the integrals on the right-hand side of (46) can be calculated analytically and we obtain

$$f(\tau) = \sum_{k=1}^{N/2} \frac{2d_{2k}}{\sqrt{B(\tau)}} \exp\left(-\frac{2}{B}\left\{|\alpha_{2k}|^2\tau[G^2 + \tau(\sigma_q + \sigma_p)] - \text{Re}\left(\alpha_{2k}^2\right)\tau^2(\sigma_q - \sigma_p)\right\}\right), \tag{48}$$

where

$$B(\tau) = [G^2(\tau) + 2\sigma_q \tau][G^2(\tau) + 2\sigma_p \tau]. \tag{49}$$

Calculating the linear term of the Taylor expansion of function $f(\tau)$ given by (48), we obtain the formula for the IDT

$$\tau_{\rm in}^{-1} = \sigma_q + \sigma_p - \sigma + 4\sum_{k=1}^{N/2} d_{2k}|\alpha_{2k}|^2 \tag{50}$$

which clearly shows that both the increase in the effective size of the superposition (the last sum on the right-hand side) or the coefficients σ_q and σ_p result in diminishing the initial decoherence time (due to the increase in the total energy of the initial state; note that $\sigma_q + \sigma_p \geqslant 1$ as a consequence of the uncertainty relation $\sigma_q\sigma_p \geqslant 1/4$). But quadratic terms (with respect to τ) in the argument of exponential function in (48) also clearly show that the rate of decay of the function $f(\tau)$ can be significantly accelerated or slowed down for $\tau > \tau_{\rm in}$, depending on signs of the difference $\sigma_q - \sigma_p$ and the quantity $\mathrm{Re}\left(\alpha_{2k}^2\right)$. Consequently, the dependence of the *final decoherence time* on the parameters σ_q, σ_p and α_{2k} can be quite complicated, so that for some combinations of these parameters the FDT can be much bigger than for others. This can occur, for example, in the case of superpositions of *squeezed* states, for which (47) is the *exact* Wigner function of the fiducial squeezed vacuum state. However, we shall not investigate this special case here.

The Gaussian approximation of the Wigner function (11) of the Fock state reads $F_m(z) \approx F_m(0)\exp[-2|z|^2(1+2m)]$. It is good for $|z| \ll \sqrt{Z_1}/2$, where Z_1 is the first zero of the Laguerre polynomial $L_m(x)$. In terms of the variable τ, this means that formula (48) can be used under the condition $\tau \ll Z_1/2$. For $m \gg 1$ we have $Z_1 \approx 2/m$, so that the domain of validity of the Gaussian approximation for the time-dependent height of the central interference peak is $m\tau \ll 1$, which implies the restriction $m \ll |\alpha_1|^2$, if one takes into account that $\tau_f \sim |\alpha_1|^{-2}$, as shown in the following section. Therefore, formula (45) cannot be derived from the Gaussian approximation.

5. Examples

To illustrate general results of the preceding sections, let us consider the symmetrical superpositions of an even number N of displaced Fock states with equal weights $d_k = 1/N$, uniformly distributed along the straight line,

$$\alpha_{2k-1} = a(2k-1), \qquad k = 1, 2, \ldots, N/2,$$

so that $2a \gg 1$ is the constant distance between the neighboring components; the maximal displacement parameter is $|\alpha_N| = a(N-1)$. Calculating the *initial decoherence time* for $m = 0$ and $\sigma = 1$, we obtain, according to equation (37),

$$\tau_{\rm in}^{-1} = \frac{2a^2}{3}(N^2-1) = \frac{2}{3}\alpha_N^2\frac{N+1}{N-1} = \begin{cases} 2\alpha^2, & N = 2 \\ \frac{2}{3}\alpha_N^2, & N \gg 1. \end{cases}$$

Consequently, if the maximal displacement $|\alpha_N|$ is fixed, then the initial decoherence time does not depend on the number of components N (for $N \gg 1$).

But this number influences the *final decoherence time* (FDT). Noting that the height of the central interference peak is N times bigger than the height of any 'constituent' peak (centered at $z = \alpha_k$) at $\tau = 0$, we can define the FDT τ_f as the solution of equation

$$f(\tau_f) = 1/(N\beta), \qquad \beta \geqslant 1, \tag{51}$$

i.e., at the instant when the central interference peak becomes β times smaller than the initial height of constituent peaks. This definition seems reasonable, because hardly one can say that the interference has disappeared, if the interference peaks remain higher than the constituent ones. The choice of β is a matter of convenience. It can be fixed, e.g., by the requirement that for the simplest superposition of only two coherent states (when $f(\tau)$ is a simple exponential function for $\sigma = 1$) the 'initial' and 'final' decoherence times coincide. Then, $\beta = \mathrm{e}/2 \approx 1.36$.

Comparing equations (40) and (51) and taking into account that the terms with $k > 1$ decrease much faster than the first one, one can see that the final decoherence time can be found from the equation

$$\frac{2r^m}{s^{m+1}} \exp\left[-\frac{2\tau}{s}|\alpha_1|^2\right] L_m\left(-\frac{4\tau^2}{rs}|\alpha_1|^2\right) = \frac{1}{\beta} \tag{52}$$

which does not contain N. For $m = 0$, we obtain a quadratic dependence of the final decoherence time on the number of components in the superposition (if τ_{in} is fixed):

$$\tau_f^{(0)}(N) = \frac{\ln(2\beta)}{2|\alpha_1|^2} = \frac{1}{3}N^2\ln(2\beta)\tau_{\text{in}}, \qquad N \gg 1. \tag{53}$$

Numerical calculations show that the final decoherence time increases with the increase of m, and the maximal value of τ_f is achieved at $m = |\alpha_1|^2$ (if $\beta > 2$):

$$\tau_f^{(m=|\alpha_1|^2)} \approx \frac{\beta^2}{2\pi|\alpha_1|^2} = \frac{\beta^2}{3\pi}N^2\tau_{\text{in}}, \qquad N \gg 1. \tag{54}$$

In all illustrations of this section, we consider the case $\sigma = 1$ (zero temperature of the reservoir), when $s \equiv 1$ and $r = 1 - 2\tau$. It can be shown that the 'reduced' FDT *in the* τ*-scale* τ_f almost does not depend on the parameter σ, including its sign. An account of the concrete value of σ gives corrections $\delta\tau_f \sim |\gamma\alpha^4|^{-1}$, which can be neglected. In the usual t-scale, the decoherence time decreases with the decrease of parameter σ (increase of the absolute temperature T), roughly speaking, as $t_f \sim \tau_f\sigma/\gamma$ (i.e., as T^{-1} in the high-temperature limit), according to equation (19) with $\tau \ll 1$. However, t_f remains finite even for an infinite temperature, if the diffusion coefficient D, defined in equation (15), is fixed. The initial decoherence time almost does not depend on m if $m \leqslant |\alpha_1|^2 \ll |\alpha_N|^2$. We do not consider the values $m \gg |\alpha_1|^2$, because they result in a strong overlap between the initial Wigner functions of the constituent components.

Formula (32) for the purity can be rewritten in the case under study as (for $m = 0$)

$$\mu = \frac{1}{N} + \frac{2}{N^2}\sum_{k=1}^{N-1}(N-k)\exp(-4\tau a^2k^2), \qquad \tau \ll 1, \tag{55}$$

so that equation (38) assumes the form

$$\sum_{k=1}^{N-1}(1-k/N)\exp(-4\tau a^2k^2) = \epsilon/2. \tag{56}$$

Again only the first term on the left-hand side should be taken into account. Consequently,

$$\tau_{f\mu} = \ln(2/\epsilon)/a^2 = \tfrac{1}{6}\tau_{\text{in}}N^2\ln(2/\epsilon), \quad N \gg 1. \tag{57}$$

To get further insight in the decoherence dynamics of multicomponent superpositions, let us consider the superpositions of four states distributed along a straight line. Figure 2 illustrates the case of equal weights of all components. In the left plot, we show the section $\mathrm{Im}(z) = 0$ of the Wigner function $F(z)$ (22) with $m = 0$ (an initial superposition of coherent states), taken at three instants of time: $\tau = 0$, $\tau = \tau_{\text{in}}$ (calculated according to equation (35)),

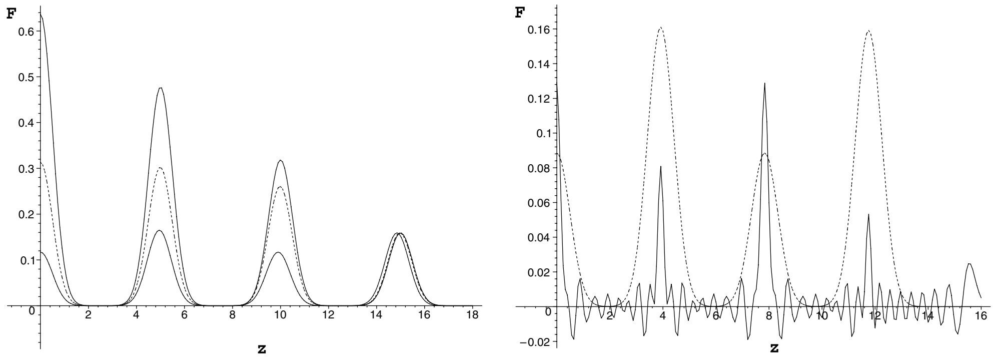

Figure 2. Left: the section $\mathrm{Im}(z) = 0$ of the Wigner function $F(z)$ (22) of the superposition of four coherent states ($m = 0$) with equal weights, located initially on the line at the points $\alpha_1 = -\alpha_2 = 5$ and $\alpha_3 = -\alpha_4 = 15$, for three instants of time: $\tau = 0$ (the upper line), $\tau = \tau_{\mathrm{in}} = 0.004$ (the middle dotted line), and $\tau = \tau_f = 0.02$ (the lower line), which corresponds to the threshold parameter $\beta = \mathrm{e}/2$. Right: the same for the superposition of four states distributed on the line ($\alpha_1 = -\alpha_2 = 4, \alpha_3 = -\alpha_4 = 12$) with equal weights, but different values of $m = 0$ (dotted line) and $m = \alpha_1^2 = 16$ (solid line), at the time instant $\tau = 0.04$.

and $\tau = \tau_f$ (calculated according to equation (51) with $\beta = \mathrm{e}/2$). The initial 'constituent' peaks are located at the points $z = \pm 5$ and $z = \pm 15$, whereas the initial peaks at the points $z = 0$ and $z = 10$ are due to the interference. The initial height of the peak at $z = 5$ is three times bigger than that at $z = 15$ due to the accidental coincidence with the interference peak formed by the pair of states with $\alpha_2 = -5$ and $\alpha_3 = 15$. Since $F(z) = F(-z)$ for the states concerned, we do not show the section of the Wigner function on the negative semi-axis. We see that although the height of the central interference peak is twice smaller than the initial one at the instant τ_{in}, it is still bigger than the heights of constituent peaks, so that hardly one can believe that an essential decoherence occurs at this instant of time. Only after time τ_f (which is five times bigger than τ_{in} in the example considered), the heights of interference peaks become smaller than those of the constituent ones.

The right plot of figure 2 illustrates the difference between superpositions of the coherent and displaced number states with big excitation numbers. The chosen instant of time $\tau = 0.04$ is 6.4 times bigger than the initial decoherence time of the superpositions of coherent states, calculated according to equation (35), but it is close to the final decoherence time (FDT) $\tau_f^{(\max)}$ of the most robust states with $m = |\alpha_1|^2$, calculated according to (54) (for $\beta = 2$). We see that the interference peaks of the Gaussian superposition are almost twice lower than the constituent peaks. At the same instant of time, the interference peaks of the highly excited non-Gaussian state are almost twice higher than the constituent ones.

To study the influence of weights of different components of the superposition on the decoherence time, we have considered superpositions of four *coherent* states with $\alpha_1 = -\alpha_2, \alpha_3 = -\alpha_4, d_1 = d_2 = x/2$ and $d_3 = d_4 = (1 - x)/2$. The initial decoherence time (35) was maintained fixed by imposing the constraint

$$x|\alpha_1|^2 + (1 - x)|\alpha_3|^2 = (2\tau_{\mathrm{in}})^{-1} = \mathrm{const.} \tag{58}$$

In figure 3, we compare the functions $L = -\ln[\mu(\tau)]$ and $F = -\ln[f(\tau)]$ for different values of the weight x and fixed values $\alpha_1 = 5$ and $\tau_{\mathrm{in}}^{-1} = 450$ (so that $\alpha_3 = 15$ for $x = 0$). We see that the purity becomes close to the constant asymptotical value $\mu_\infty = [x^2 + (1 - x)^2]/2$ after the time interval which is significantly greater than the IDT τ_{in}, even for small values of x. The difference between the 'initial' and 'final' decoherence times is even more pronounced, if one

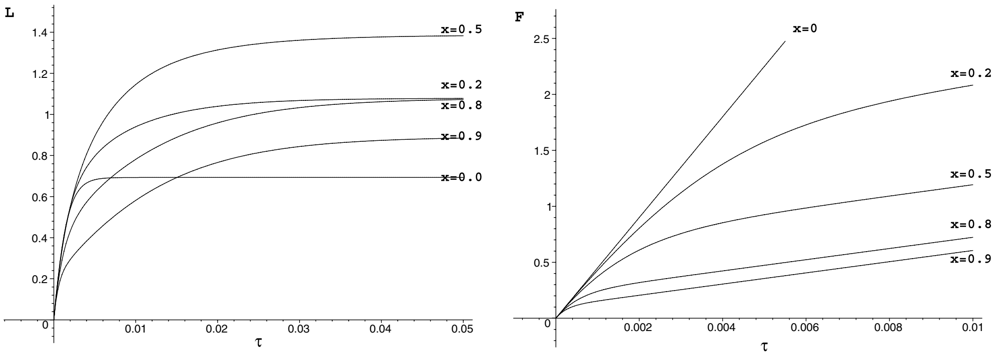

Figure 3. The inverse logarithm of purity $L = -\ln(\mu)$ (at the left) and the inverse logarithm of the normalized height of the central interference peak $F = -\ln(f)$ (at the right) versus the dimensionless time τ for the superpositions of four coherent states on the line with different weights of 'internal' components x. The fixed parameters are the initial decoherence time $\tau_{\text{in}} = 1/450$ and the initial position of the first component $\alpha_1 = 5$: see equation (58).

considers the time dependence of the inverse logarithm of the normalized height of the central interference peak $F = -\ln(f)$. Although all curves start with the same initial slope at $\tau = 0$, corresponding to the initial decoherence time τ_{in}, this parameter has nothing in common with the real decoherence time, except for the case of very small weight x of 'internal' components of the superposition (less than 0.1, if the threshold parameter β in equation (53) is greater than e/2).

6. A long-time behavior of the Wigner function in the case of amplification

If parameter σ defined in (15) is negative, then we have the case of a phase-insensitive amplifier [81]. The evolution of nonclassical states and their statistical properties in this case was considered by several authors [82]. Time-dependent Wigner functions and other quasi-distributions for some initial states were calculated in [83, 84]. In particular, Agarwal and Tara [83] showed how phase-insensitive amplifiers can be used to measure different quantum quasi-distribution functions. The phase-insensitive amplification of two-component quantum superpositions of the form (1) was considered in [20, 85–87]. The evolution of the Wigner function for initial superpositions of displaced Fock states was studied in [88]. Here we give new results for initial superpositions of a more general form (4). Namely, we consider the asymptotical behavior of the Wigner function (22) for $t \to \infty$ in the case of amplification.

If $0 > \sigma > -1$, then asymptotically

$$G = \exp(|\gamma|t) \gg 1, \qquad \tau \approx G^2/|\sigma|,$$
$$r \approx -G^2(1-|\sigma|)/|\sigma| < 0, \qquad s \approx G^2(|\sigma|+1)/|\sigma|.$$

Consequently, one can rewrite function (22) for $\tau \gg 1$ as

$$F(z;t) = G^{-2}(\tau)\tilde{F}(\tilde{z}), \qquad \tilde{z} = z/G(t), \tag{59}$$

where

$$\tilde{F}(\tilde{z}) = \frac{2|\sigma|(1-|\sigma|)^m}{A\pi(1+|\sigma|)^{m+1}} \sum_{j,k=1}^{N} c_j c_k^* L_m\left(-\frac{4}{1-\sigma^2} Z_{jk}^{(+)} Z_{jk}^{(-)*}\right) \exp\left[-\frac{1}{2}(|\alpha_j|^2 + |\alpha_k|^2) - \frac{2|\sigma|}{1+|\sigma|}|\tilde{z}|^2 + \frac{1-|\sigma|}{1+|\sigma|}\alpha_j\alpha_k^* + \frac{2|\sigma|}{1+|\sigma|}\left(\tilde{z}\alpha_k^* + \tilde{z}^*\alpha_j\right)\right], \tag{60}$$

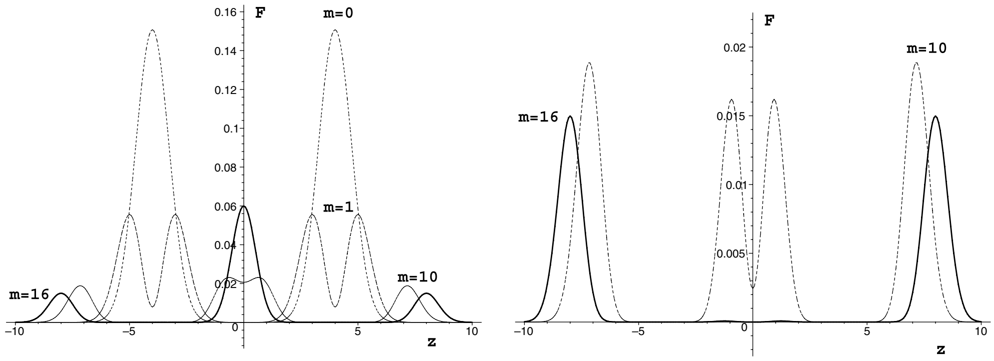

Figure 4. Left: the section $\mathrm{Im}(z) = 0$ of the scaled Wigner function $\tilde{F}(\tilde{z})$ (60) versus the scaled argument $\tilde{z}$ (tildes over z and F are omitted) in the case of amplification ($\sigma = -0.9$) of the initial 'constructive' superposition of two displaced number states with $c_1 = c_2 = 1/\sqrt{2}$ and $\alpha_1 = -\alpha_2 = 4$, for different values of $m = 0, 1, 10, 16$. A solid line for $m = 16$ has a strong peak at the center, whereas the dotted line for $m = 10$ has a lower double peak at the center. Right: the same for the initial 'destructive' superposition of two displaced number states with $c_1 = -c_2 = 1/\sqrt{2}$ and $\alpha_1 = -\alpha_2 = 4$, for two values of $m = 10, 16$. A solid line for $m = 16$ shows no peak at the center.

$$Z_{jk}^{(\pm)} = |\sigma|[\tilde{z} - \tfrac{1}{2}(\alpha_j + \alpha_k)] \pm \tfrac{1}{2}(\alpha_j - \alpha_k). \tag{61}$$

We see that the asymptotical Wigner function takes a 'frozen' form in terms of the scaled variable $\tilde{z}$. The time-dependent factor $G^{-2}(\tau)$ in equation (59) is responsible for the correct normalization, due to the relation $\mathrm{d}^2 z = G^2 \mathrm{d}^2 \tilde{z}$. On the other hand, since the function $F^2(z)$ decays with time as $G^{-4}(\tau)$, the quantum purity (27) decays as $G^{-2}(\tau) = \exp(-2|\gamma|t)$.

Nonetheless, the contribution of the off-diagonal terms in (60) cannot be neglected when $\tau \gg 1$, because the absolute values of arguments of the Laguerre polynomials on the right-hand side are much bigger than the absolute values of arguments of the corresponding exponential functions with the same values of variable $\tilde{z}$ if σ is close to -1. This fact can be interpreted as a 'revival' of interference effects (the asymptotical Wigner function depends on the phases of coefficients c_j, and not on their absolute values only). It is clearly seen if one compares two plots of figure 4, which show the sections of the frozen Wigner function $\tilde{F}(\tilde{z})$ with $\tilde{z}$ real for even and odd initial superpositions of two displaced number states with different values of parameter m. For $m = 0$ and $m = 1$, the curves for even and odd initial states practically coincide, so these values of m are not considered in the right plot. But when m is close to $|\alpha|^2$, the difference becomes striking. For example, the central interference peak of the even state with $m = 16$ is totally destroyed in the case of an odd superposition.

However, this interference has a *classical* nature, not a quantum one. In the most distinct form, this phenomenon can be seen in the limit case of an *ideal quantum amplifier* characterized by the parameter $\sigma = -1$ (this value corresponds to the 'negative zero temperature'). Then, $\tau \approx G^2$ and $s \approx 2\tau$, but $r \equiv 1$, so that function (22) has the asymptotical form:

$$\begin{aligned} F(z;\tau) = \frac{2(-1)^m}{A\pi(2\tau)^{m+1}} \sum_{j,k=1}^{N} c_j c_k^* L_m(2\tau[\tilde{z} - \alpha_k][\tilde{z} - \alpha_j]^*) \\ \times \exp[-|\tilde{z}|^2 + \tilde{z}\alpha_k^* + \tilde{z}^*\alpha_j - (|\alpha_j|^2 + |\alpha_k|^2)/2]. \end{aligned} \tag{62}$$

Under the condition

$$2\tau|(\tilde{z} - \alpha_k)(\tilde{z} - \alpha_j)^*| \gg m^2, \tag{63}$$

one can replace the Laguerre polynomial by the highest-power term in the sum (12), writing $L_m(x) \approx (-x)^m/m!$. Then, function (62) becomes

$$F(z;\tau) = (A\pi\tau m!)^{-1} \sum_{j,k=1}^{N} c_j c_k^* [(\tilde{z}-\alpha_k)(\tilde{z}-\alpha_j)^*]^m \\ \times \exp[-|\tilde{z}|^2 + \tilde{z}\alpha_k^* + \tilde{z}^*\alpha_j - (|\alpha_j|^2 + |\alpha_k|^2)/2]. \tag{64}$$

It is remarkable that the right-hand side of equation (64) can be represented in a *factorized form* as the modulus squared of some 'Wigner wavefunction':

$$F(z;\tau) = \Psi(z;\tau)\Psi^*(z,\tau), \tag{65}$$

$$\Psi(z;\tau) = (A\pi\tau m!)^{-1/2} \sum_{j=1}^{N} c_j (\tilde{z}-\alpha_j)^{*m} \exp\left[-\tfrac{1}{2}|\tilde{z}-\alpha_j|^2 + \mathrm{i}\,\mathrm{Im}(\tilde{z}^*\alpha_j)\right]. \tag{66}$$

Equation (65) shows that the asymptotical Wigner function is positive at all points of the phase space. This means that the asymptotical state is *totally classical*.[1] On the other hand, equation (66) shows that this classical state is very sensitive to the relative phases of coefficients c_j. It is worth mentioning that function (66) 'remembers' not only the exact values of coefficients of the initial *quantum* superposition (4), but also it preserves some memory of the initial form of the 'constituent' wave packets through the power m of pre-exponential factors.

7. Conclusion

The main results of the paper are as follows. We have obtained a simple formula (7) for the Wigner function of a superposition of an arbitrary number of states generated by displacement operators from an arbitrary given 'fiducial' state, in terms of the Wigner function of this fiducial state. We have obtained an exact formula (22) describing the evolution of the Wigner function of a superposition of an arbitrary number of displaced number states with the same excitation number m, governed by the standard master equation (13), both in the case of a relaxation to a thermal state with an arbitrary temperature and in the case of a phase-insensitive quantum amplifier.

We compared two definitions of the decoherence time: one based on the time evolution of the quantum purity (μ-decoherence time) and another based on the time evolution of the central interference peak (f-decoherence time). We showed that both approaches give identical results for the 'initial decoherence time' (IDT) in the case of symmetrical superpositions. We demonstrated that the IDT can be considered as a reliable characteristic of the decoherence process only in the simplest case of superpositions of two coherent states. In more involved cases, such as superpositions of more than two well-separated 'constituent' packets or packets with some 'internal structure' (represented by displaced number states), the concept of 'final decoherence time' (FDT) was introduced. The 'f-approach' to calculating the FDT is much more simple and transparent than the 'μ-approach', because the latter can be used, as a matter of fact, only for superpositions of initial coherent states. For more involved superpositions, the calculation of quantum purity at any instant of time becomes an extremely difficult problem, even if the explicit expression for the Wigner function is known. In contrast, the analysis of

[1] There is no contradiction with the results of [85], according to which any initial nonclassical state maintains some degree of nonclassicality for any time in the process of amplification: simply this degree of nonclassicality becomes asymptotically exponentially small, because the Wigner function can assume small negative values only in domains where inequality (63) is not fulfilled, and the total area of these domains goes asymptotically to zero.

the time evolution of the central interference peak can be easily performed for any value of the time variable and for a rather arbitrary (symmetrical) fiducial state $|g\rangle$, according to the approximate formula (48). It was shown that although the μ-FDT and f-FDT do not coincide in a generic case, their qualitative behavior is similar. Both the times *increase* significantly (for several times or even by orders of magnitude), if one increases the number of constituent states N or the excitation number m (in the case of displaced number states), maintaining the value of the IDT. Therefore, more involved superpositions can be more robust against the decoherence than simple superpositions of two coherent states (Gaussian packets). Finally, we have shown how initial *quantum* superpositions are transformed into *classical* superpositions in the case of phase-insensitive amplification.

We did not touch many interesting subjects, such as the decoherence caused by different types of reservoirs (e.g., phase-sensitive ones or those describing the phase damping), the decoherence in systems with many degrees of freedom (modes), relations between the purity, height of interference peak of the Wigner function and the evolution of the off-diagonal matrix elements of the statistical operator in different bases (e.g., in the discrete Fock basis) and so on. The related results will be reported elsewhere.

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